“Study of Existence and Uniqueness Solutions of Fractional Order Differential and Difference Equations and its Applications”

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Introduction

In the middle of the nineteenth century, fractional calculus was created as a mathematical theory. Fractional differential equations can be used to more precisely model physical systems rather than conventional differential equations compared to the traditional differential equation. The advantages have been established by a large amount of research. Fractional calculus is used to model and regulate many dynamical systems, developing numerical approaches for fractional differential equations is an important subject of research. Several strategies for generating numerical solutions for fractional differential equations have been presented in recent years. S. Das (2009) used the vibrational iteration approach to get an analytical solution to the fractional diffusion problem.

Difference equations are meant to describe some property or phenomenon that change over time. In contrast to differential equations, difference equations deal with discrete time or sequential discrete free variables in general. Many problems are discrete by nature like population size over successive generations, machine state over several iterations, or a physical phenomenon in successive time instances or it can be used in an approximation of a continuous phenomenon for easier analysis and the possibility of using computer simulations, etc. Difference equations can prove to be an extremely powerful (and sometimes the only) tool in handling many complex problems. They find it used frequently in stability, asymptotic, oscillation theories, and system control theory. In turn, these fields affect many other disciplines from biology and economy to physics and computer science. The Elaydi [1] introduces the idea of difference equations as well as a number of tools and several application examples. V. Lakshmikantham [2] introduce iterative processes and numerical methods for difference equations and theory of stability of Lyapunov including converse theorems and total and practical stability. Moreover, discussed the application of the theory of difference equations to numerical analysis.

Fractional order calculus has received recent attention as an alternative calculus in mathematics and has been used in a number of scientific and technical fields. Recently, the application of fractional calculus modelling is developed to several domains of research, including as rheology, biology, and geomorphology. Indeed, it is generally understood that the introduction of fractional derivatives provides the system with a critical memory mechanism. It is therefore important to understand the advantages of using fractional derivatives in classical equations. Currently, several different methods are used to solve fractional differential equations

I. Preliminaries

Fractional calculus is one of many applied mathematics fields. It deals with arbitrarily ordered derivatives and integrals as well as their applications in place of integer order solutions to fractional differential and difference equations. Some important fractional derivatives are as below

a) Riemann Liouville Fractional Derivative :

For $\alpha \in [n-1, n)$, then Riemann Liouville Fractional Derivative of f of order α is defined as

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

b) Caputo's Fractional Derivative :

Caputo definition.

For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}} dx$$

The notation of fractional difference in the sense of Caputo for the different notions of fractional sums

Let $\alpha > 0$ and $a \in \mathbb{R}$. the α-th fractional sum of a function f is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

Where $t \in \mathbb{N}_{a+\alpha}$ and $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

c) Atici and Eloe Fractional Difference :

The following definition was proposed by Atici and Eloe (2009). Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s-1)^{\overline{\alpha-1}} f(s),$$

Where $t \in \mathbb{N}_{a+\alpha}$ and $t^{\overline{(\alpha)}} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

The following definition of the (backward nabla) discrete fractional sum was proposed by Grey and Zhang in 1998.

The (nabla) left fractional sum of $\alpha > 0$ (starting from a) is defined by

$${}_{GZ}\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s-1)^{\overline{\alpha-1}} f(s),$$

Where $t \in \mathbb{N}_{a+1}$.

Let $\alpha > 0$ and α-th fractional Riemann-Liouville like difference is defined by

$$\Delta_a^{-\alpha} f(t) = \Delta_{a+(m-\alpha)}^{m}\left(\Delta_a^{-(m-\alpha)} f\right)(t), \quad t \in \mathbb{N}_{a+(m-\alpha)},$$

Where $m-1 < \alpha < m, m = \lceil\alpha\rceil$. Here $\lceil . \rceil$ denotes the ceiling number i.e the smallest interger not less than α.

Let $f: \mathbb{N}_0 \to X$ be a vector-valued sequence and $\alpha > 0$. The α-th fractional sum of f, denoted by $\Delta^{-\alpha} f(n) \coloneqq \sum_{j=0}^{n} k^{\alpha}(n-j) f(j)\,,\ n \in \mathbb{N}_0$,

Where

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \qquad j \in \mathbb{N}_0$$

.

The function $\Delta^{-\alpha}: (\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ is called the fractional delta operator.

Now, we present the definition of fractional differences. The next concept, given by Miller and Ross in 1989, is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville.

d) Riemann-Liouville fractional Difference:

Let $\alpha > 0$. The α-th fractional Riemann-Liouville like difference is defined by

$$\Delta_a^{\alpha} f(t) = \Delta_{a+(m-\alpha)}^{m}\left(\Delta_a^{-(m-\alpha)} f\right)(t),\ t \in \mathbb{N}_{a+(m-\alpha)},$$

Where $m-1 < \alpha < m, m = \lceil\alpha\rceil$.

Analogously, following the sense of Riemann-Liouville described above, the next definition appears given by Abdeljawad and Atici in 2012

The nabla (left) fractional difference operator of order $\alpha > 0$ is defined by

$${}_{GZ}\nabla_a^{-\alpha} f(t) = \Delta_{a+1}^{m}\left({}_{GZ}\nabla_a^{(m-\alpha)}\right)(t),\ \ t \in \mathbb{N}_{a+1}$$

Where $m-1 < \alpha < m, m = \lceil\alpha\rceil$.

II. Review of literature and development in the subject area

The history of fractional derivative begins in 1695 when L'hospital asked Leibniz that what meaning could be ascribed to $D^n f$ if n=1/2 in his letter, Leibniz replied it would be a paradox.after that the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development. They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. Lacroix was first mathematician who released paper mentioning fractional derivatives in it. He began with the polynomial $(x) = x^m$, where m is a positive integer, and differentiated it n times where $m \geq n$ to , get $\frac{Df}{Dx} = \frac{m!}{(m-n)!}$ then he used Legendre's symbol Γ to have $\frac{D^n f}{Dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x\hat{}(m-n)$ Using this formula when $m = 1$ and $n = \frac{1}{2}$ he obtained $d^{\frac{1}{2}} f = \frac{2\sqrt{x}}{\pi}$.

Firstly, Miller and Ross and then Gray and Zhang introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives.

The time differences of fractional order were initially examined by Kuttner. A discrete fractional difference operator that is specified as an infinite series was first introduced by Diaz and Osler. A fractional calculus was created by Grey and Zhang for the discrete nabla difference operator. Miller and Ross defined a fractional sum at the same time by solving a linear difference equation. The Riemann-Liouville-like fractional difference was invented by Atici and Eloe, who also introduced it. These qualities make it possible to solve specific fractional difference problems. Ferreira introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. Holm developed and applied the tools of discrete fractional calculus to the area of fractional difference equations. Abdeljawad obtained dual identities in fractional difference calculus which they relate the delta and nabla and the left and right fractional sums and differences. In the area of discrete fractional calculus, Goodrich and Peterson derive fundamental theoretical conclusions. Additionally, Goodrich investigated geometrical features and the presence of positive solutions for discrete fractional systems.

In 2009, Malik Muslim have studied the fractional order semilinear differential equation in an arbitrary Banach space x. They **used** the analytic semi group theory of linear operators and fixed point method to establish the existence and uniqueness of solutions of the given problem. They also proved the existence of a global solution. Existence and convergence of an approximate solution to the given problem is also proved in a separable Hilbert space.

In 2010, Bashir Ahmad Alsaedi solved some initial value problem for a coupled Caputo type nonlinear fractional differential system of higher order. As a first problem,they exhibited that the non-homogeneous terms in the coupled fractional differential system depend on the fractional derivatives of lower orders only. Then the non-homogeneous terms in the fractional differential system are allowed to depend on the unknown functions together with the fractional derivative of lower orders. Their method of analysis is based on the reduction of the given system to an equivalent system of integral equations. Applying the nonlinear alternative of Leray-Schauder, they prove the existence of solutions of the fractional differential system. The uniqueness of solutions of the fractional differential system is established by using the Banach contraction principle. Some illustrative examples are also given.

In 2013, Bin Zheng, Chuanbao Wen have proposed a new fractional sub-equation method for finding exact solutions of fractional partial differential equations (FPDEs) in the sense of modified Riemann-Liouville derivative, which is the fractional version of the known (G'/G) method. To illustrate the validity of this method, they applied it to the space-time fractional Fokas equation, the space-time fractional -dimensional dispersive long wave equations and the

space-time fractional fifth- order Sawada-Kotera equation. As a result, some new exact solutions for them are successfully established

In 1993, Parlange etal described the one dimensional diffusion equation for soil moisture, Parlange has also worked on other topics related to soil moisture including horizontal infiltration of water in soils solving the flow equation in unsaturated soils by optimization etc.

In 2013, S. M. Jogdand and K. C. Takale have develop the implicit finite difference scheme for space fractional soil moisture diffusion equation (SFSMDE) with initial and boundary conditions. They prove that the scheme is unconditionally stable and convergent.

In 2014, Pholakrit Thiramaus, Sotiris K Ntouyas, Jessada Tariboon have studied the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and non-local fractional integral boundary conditions. The results are based on some classical fixed point theorems.

In 2018,Hossein Fazil,Juan J Nieto,Fariba Bahrami devoted to proving the existence, uniqueness and smoothness of solutions for differential equations with Riemann-Liouville sequential fractional derivatives admitting only the existence of a lower solution. The appropriate fractional Sobolev spaces equipped with a suitable partial order are introduced, and applying some fixed point theorems the results are proved. The approximate solutions are presented to the corresponding initial value problem

In 2019. Dnyanoba Dhaigude, Bakr Rizqan develop a monotone iterative technique by introducing upper and lower solutions to Riemann- Liouville fractional differential equations with deviating arguments and integral boundary conditions. As an application of this technique, existence and uniqueness results are obtained.

In 2022, Vidushi Gupta, NatrajanValiammal, Chokkalingam Ravichandrean, kottakaran sooppy Nisar, Fahd Jarad have studied the investigation of existence and uniqueness of impulsive dynamical fractional systems with quadratic perturbation of second type subject to non-local boundary conditions is presented and proved. By employing the fractional theory, Banach contraction technique, and Krasnoselskii's fixed point theorem, they derived some sufficient conditions to ensure the existence of the system.

III. Objectives of Proposed Research Work:

1) To study basic concept of fractional order differential and difference equations.
2) Analysis of different methods for differential and difference equations which are available in the various fields.
3) To study basic concept of fractional calculus.

4) To obtain existence and uniqueness solutions of fractional order differential and difference equations

5) To analyze the system of Mathematical model of differential equations for fractional order.

6) To publish the scientific papers and expected to be published in international journals indexed in UGC Care listed journal.

IV) Research Methodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on differential and difference equation and fractional partial differential equation.

2) To get the results by reviewing latest research in the area

3) To study Iterative / Numerical methods for obtaining approximate solutions of fractional differential equations.

4) To represent numerical solutions of fractional differential and difference equations graphically by Mathematica, Matlab, Python, Scilab, Mathcad and discuss their nature.

V) Importance of the Study:

Fractional calculus is becoming a useful and key tool in the analysis of

scientific problems in Diffusion Processes, Soil moisture study, Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, integration and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under the curve.

a) International Status: Many scientists and researchers contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache .

Lynn R. Willams (1979) proved the existence of multiple fixed point theorems of positive, completely continuous operators on functional spaces which are used in chemical reactor theory. Nguyen Huu Denn (1993) proved a fixed common point theorem for the pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space. He also investigated the class of compactible mappings and establish some fixed point theorems for these mappings on the Meir-Keeler condition. Blazej Szmanda (1996), obtained criterion for the oscillatory and asymptotic behaviour of solutions of nonlinear difference equation.

b) National Status: Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar, Sunny Chauhan and Pant. S.P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R.P. Pant from Nainatal, D.N Chaliszar from Gujrat, and many more from India had worked on fixed point theorems and its various applications.

c) Regional Status: Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like M. Kolwankar, A. K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji , J. Verma, D. B. Dhaigude.

c) Scope of Study:

1) In recent years, there has been a great deal of interest in fractional differential and difference equations. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. There are few iterative methods for obtaining the solution of fractional differential and difference equation (FDE), therefore, we develop new existence and uniqueness results to obtain their solutions and discuss the stability and more results.

2) The fractional calculus finds use in many fields of science and engineering including the fluid flow, quantitative biology, electro-chemistry, scattering theory, diffusion transport theory, chemical physics and statistical probability theory, potential theory and many branches of mathematical analysis, like integral and differential equations, operational calculus and univalent function theory, dynamical systems theory, tuning of PID controllers, heat diffusion, circuit synthesis etc. Virtually no area of classical and modern analysis is left untouched by fractional calculus.

3) A significantly large number of earlier works on the subject of fractional calculus give an interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis like ordinary and partial differential equations, integral equations, special functions, summation of series, etc.

4) The fractional integral operators involving various special functions, have found significant importance and applications in various sub-field of applicable mathematical analysis.

Expected Outcome:

1) We will try to analyze the development in new results and iterative methods for fractional differential and difference equations.

2) We will try find out existence and uniqueness solutions of fractional differential and difference equations

3) To find the approximate solutions of fractional differential and difference equation by old and new technique and it will represented graphical by different software's like Mathematica, Matlab, Python, Scilab, Mathcad etc.

VI) Purpose of carrying out proposed research work:

The purpose of the research is study of existence and uniqueness solutions of fractional order differential and difference equations using some fixed point theorems. The solution of fractional differential equations will be obtained by new methods using mathematical software. Also we will study Numerical methods to obtain the solutions of fractional differential and difference equations. We will compare the solutions obtain by Analytical methods and Numerical methods. The results will be represented graphically

Year wise plan of work and targets to be achieved:

Sr. No	Period	Plan of work	Targets to be Achieved
1	First year	Study of background information relevant to the research issue. To survey the number of reputed research papers in fractional calculus. To visit reputed libraries	Good command on the subject and clear the course work
2	Second year	Collecting and analyzing research articles. To attend national, International Conferences. and presentation of research papers	Try to get some new results based on the previous work and try to submit some research papers to reputed journals.
3	Third year	To develop and new approach for obtain the solution of fractional differential and difference equations. Publishing some more papers and writing of the thesis.	Published at least two papers.
4	Fourth year	Submitting the thesis	Some more papers and Thesis submission

VII) References:

1) S. Elaydi, An Introduction to Difference Equations, Second Edition, Springer New York, NY, 1999.

2) V. Lakshmikantham, D. Trigiante, Theory of Difference Equations Numerical Mehtods and Applications, First Edition, 1988.

3) R. Agrawal, Difference equations and inequalities, Marcel Dekker, New York, 1992.

4) Kelley W. and Peterson A., Difference Equations: An Introduction with Applications, Second Edition, Academic Press, New York, New York, 2000.

5) F. Mainardi, "Fractional Calculus: Theory and Applications," Mathematics, vol. 46, Pp. 4, 2018.

6) J. B. Diaz and TJ Olser, Differences of Fractional Order, Mathematics of Computation, 28, 1974.

7) H Gray and N Zhang, On a New Definition of the Fractional Difference, Mathematics of Computation, 50, 513-529.

8) I. Podubny Fractional Differential Equations. Academics press, New York, 1999.

9) S. Momani Analytic approximate solutions for fractional heal-like and wave-like equations with variable co-efficients using the decomposition method. Appl. Math. Comput. 165 (2005)459-472.

10) J. H. He, A new approach to linear partial differential equation, common. Nonlinear Sci. Numer. Simul. 2 (4) (1997)230-235.

11) S. Bhalekar, V Daftardar–Gejji Solving evolution equations using a new iterative method. Numer. Methods, partial differential equations. 26 (4) (2010)906-916.

12) A.A. Kilbas, H.M. Srivastva and J.J. Trajillo, Theory and applications of fractional differential equations.

13) R. Hilfer, Application of fractional calculus in Physics, world scientific Singapore (2000).

14) E. Scalas, R. Gorenflo and F. Mainardi, Fractional Calculus and Continuous – time finance, PhysicaA 284(2000).

15) T. Abdeljawad. On Riemann and Caputo fractional differences. Comput. Math. Appl. 62, 1602-1611, 2011.

16) C. S. Goodrich. Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. 61, 191–202, 2011
17) S. M. Jogdand, K.C. Takale, V.C. Borkar, Fractional Order Finite Difference Scheme For Soil Moisture Diffusion Equation and its Applications, IOSR Journal of Mathematics 5, 12-18, 2013
18) Y. Cui, "Uniqueness of solution for boundary value problems for fractional differential equations,"Applied Mathematics Letters, vol. 51, pp. 48-54, 2016.
19) Hossein Fazli, Juan J Nieto, Fariba Bahrami On the existence and uniqueness results for nonlinear sequential fractional differential equations Appl. Comput. Math 17(1),36-47, 2018
20) Dnyanoba Dhaigude, Bakr Rizqan Existence and uniqueness of solutions of fractional differential equations with deviating arguments under integral boundary conditions Kyungpook Mathematical Journal 59 (1), 191-202, 2019.
21) Sunil Kumar, Amit Kumar, Syed Abbas, Maysaa Al Qurashi, Dumitru Baleanu A modified analytical approach with existence and uniqueness for fractional Cauchy reaction- diffusion equations Advances in Difference Equations 2020 (1), 1-18,2020
22) V Daftardar-Gejji, P Prakash, S Choudhary, Exact solutions of generalized nonlinear time-fractional reaction–diffusion equations with time delay, The European Physical Journal Plus 135, 1-24, 2021
23) Vidushi Gupta, Fahd Jarad, Natarajan Valliammal, Chokkalingam Ravichandran, Kottakkaran Sooppy Nisar, Existence and uniqueness of solutions for fractional nonlinear hybrid impulsive system Numerical Methods for Partial Differential Equations 38 (3), 359-371, 2022
24) K. C. Takale, K Ghode, S Gaikwad, Traveling Wave Solutions of Fractional Differential Equations Arising in Warm Plasma, Baghdad Science Journal 20 (1 (SI)), 0318-0318, 2023
25) S. M. Jogdand, L. C. Takale, Finite Difference Scheme for Space Fractional Diffusion Equation with Mixed Boundary Conditions, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728. Volume 5, Issue 4 (Jan. - Feb. 2013), PP 12-18
26) S. M. Jogdand, L. C. Takale, Fractional Order Finite Difference Scheme For Soil Moisture Diffusion Equation And Its Applications, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 8, Issue 4 (Sep. - Oct. 2013), PP 01-08

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